From Quadratic Variation to Cross Variation

§0 Recall that a function $f : [0,T] \to \mathbb{R}$ has variation,

$$V^1_\pi(f) \equiv V^1_\pi(f : [0,T])$$

over a partition, π, of $[0,T]$, defined by

$$V^1_\pi(f) = \sum_\pi |f(t_i) - f(t_{i-1})| .$$

We say f is of bounded variation over $[0,T]$ if

$$V^1(f) = \sup_{\pi, |\pi| \to 0} V^1_\pi(f) < \infty .$$

For a number, $p \in [1, \infty)$, the 'p-th' variation of f over $[0,T]$ is,

$$V^p(f) = \sup_{\substack{\pi \\ |\pi| \to 0}} \sum_\pi |f(t_i) - f(t_{i-1})|^p .$$

The <u>mesh</u> of π, is $\max_\pi \{ t_i - t_{i-1} \} \triangleq |\pi|$.

<u>Proposition 0.01</u> Let f be continuous and of bounded variation over $[0,T]$. Then $V^2(f) = 0$.

<u>Pf</u> f continuous, $[0,T]$ compact, then f is uniformly continuous, so for $\epsilon > 0$ $\exists \delta > 0$: $|s-t| < \delta$ then $|f(s) - f(t)| < \epsilon$ for all $s,t \in [0,T]$. Choose π with $|\pi| < \delta$, then $|f(t_i) - f(t_{i-1})| < \epsilon$ for successive points in π. And;

$$V_\pi^2(f) = \sum_\pi |f(t_i)-f(t_{i-1})|^2$$

$$\leq \sum_\pi |f(t_i)-f(t_{i-1})| \cdot \max_\pi |f(t_i)-f(t_{i-1})|$$

$$\leq \epsilon \, V_\pi^1(f)$$

Taking the sup of π with $|\pi| \to 0$ shows $V_\pi^2(f) \leq \epsilon$.

<u>Exercise:</u> Find a continuous function which is not of bounded variation on some $[0,T]$, $T>0$.

<u>Remarks.</u>

(i) functions of bounded variation are very important in finance.

(ii) a process is of bounded variation if $\mathbb{P}$ almost every path is of bounded variation.

(iii) A result, due to Lebesgue I think, states that f is of bounded variation on $[0,T]$ iff it is the difference of two increasing functions. Since every increasing function is measurable — because $f^{-1}(-\infty, t]$ i.e. either an open or a half-open interval — then every bounded variation function is measurable. Moreover it will have only countably many jumps, because this is so for increasing functions.

Stieltjes Integration

Let $g : [0,T] \to \mathbb{R}$ be bounded and $f : [0,T] \to \mathbb{R}$ of bounded variation. Let π be a partition of $[0,T]$. A Riemann Stieltjes sum of g w.r.t. f over π is,

$$\sum_{\pi} g(s_i)\left(f(t_i) - f(t_{i-1})\right)$$

where $s_i \in [t_{i-1}, t_i]$. If there is a number, α, such that, for every $\epsilon > 0$ there is a $\delta > 0$ such that for partitions π, with $|\pi| < \delta$, and any choice of $s_i \in [t_{i-1}, t_i]$, we have

$$\left| \sum_{\pi} g(s_i)\left(f(t_i) - f(t_{i-1})\right) - \alpha \right| < \epsilon$$

then we say g is Riemann-Stieltjes integrable with respect to f over $[0,T]$ and we write

$$\alpha = \int_0^T g(s)\,df(s) .$$

The "g's" for which this works are quite numerous. For example every continuous g is R-S integrable with respect to f.

When we have a <u>process</u> of bounded variation, (A_t), a <u>process</u> with suitable paths, (C_t), we can form

$$\int_0^T C_s\,dA_s \equiv \int_0^T C_s(\omega)\,dA_s(\omega)$$

This is one form of Stochastic Integration.

However, you have already seen that the paths of BM are NOT of bounded variation and so stochastic integration with respect to (W_t) CANNOT be performed as above! Strangely enough it is a process of bounded variation which allows us to "get around" this difficulty. For BM, (W_t), it is the quadratic variation, $\langle W \rangle_t$, which is simply $t I_\Omega$.

The quadratic variation of BM can be obtained as the limit over partitions, θ of $[0,T]$ of sums of the form

$$\sum_{\theta} \Delta W_{t_i}^2 \equiv \sum_i (W_{t_i} - W_{t_{i-1}})^2$$

where $\theta = \{t_i\}$ is a partition of $[0,T]^{\dagger}$

This limit over partitions occurs in L^2 norm and $\mathbb{P}$ a.s. For a general L^2 martingale, X, we can define the same type of sums;

$$\sum_{\theta} \Delta X_{t_i}^2 .$$

Evidently, this is a non-negative random variable. If we have $T_1 < T_2$, so that $[0,T_1] \subseteq [0,T_2]$, then any partition θ^1 of $[0,T_1]$ can be enlarged to obtain a partition θ^2 of $[0,T_2]$. You just throw in T_2 and any points you care to choose in (T_1, T_2) — finitely many mind. Because θ^2 generates more quadratic terms;

$$\sum_{\theta^1} \Delta X_{t_i}^2 \leq \sum_{\theta^2} \Delta X_{t_i}^2 .$$

Hold that thought, we'll need it later.

Return to $[0,T]$ and our sums over the partition θ:

$\dagger$ A finite subset of $[0,T]$ containing $0, T$.

One can show that the limit over partitions of $[0, T]$ of sums of the form

$$\sum_{\theta} \Delta X_{t_i}^2$$

will converge in probability (✱) to a random variable, $\langle X \rangle_T$. The details of the proof will not be relevant in this course, you need only to know the result.

Theorem 0.0:

So we are going to accept, without proof, that for every martingale, (X_t), with $X_t = M_t(X_\infty)$, $X_\infty \in L^2$,

$$\lim_{\theta} \sum_{\theta} \Delta X_{t_i}^2 = \langle X \rangle_T \quad \text{exists as a}$$

limit, over partitions θ, of $[0, T]$, in proba-bility.

The process $T \mapsto \langle X \rangle_T$ is adapted – think of the sums..., is increasing – recall our inital discussion, and is called the quadratic variation of X.

This result is true for a general L^2 martingale living on a stochastic base,

$$(\Omega, \mathcal{F}_\infty, [0, \infty], (\mathcal{F}_t), \mathbb{P})$$

which satisfies the "usual conditions". The existence of the quadratic variation allows us to define stochastic integrals with respect to a general L^2 martingale.

(✱) Meaning that, $\forall \epsilon > 0$ $\lim_{\theta} \mathbb{P}\{\omega : |\sum_{\theta} \Delta X_{t_i}^2 - \langle X \rangle_T| > \epsilon\} = 0$.

Here, $\lim_{\theta}$ means "as $\#\theta \to \infty$ and $\max_i |t_i - t_{i-1}| \to 0$".

We will restrict our attention to continuous martingales, i.e. those that are $\mathbb{P}$- a.s. pathwise continuous. For this it is enough that our filtration is left continuous;

$$\mathcal{F}_t = \sigma\left(\bigcup_{s<t} \mathcal{F}_s\right) ;$$

in addition to the right continuity which comes with the "usual conditions".

You already know that if f is a measurable adapted process for which

$$\mathbb{E}\left(\int_0^t |f(s,\omega)|^2 ds\right) < \infty \quad , \quad t \in \mathbb{R}^+$$

then one can form $\int_0^t f(s)\, dW_s$, $t \in \mathbb{R}^+$, and that the process,

$$t \longmapsto \int_0^t f(s)\, dW_s \equiv X_t$$

is an L^2 martingale. Sometimes we will be required to form stochastic integrals of the form $\int_0^t g(s)\, dX_s$, for suitable integrands. The quadratic variation of X, $(\langle X \rangle_t)$, shows us how to do this and the following result is fundamental. Once again we will just accept this result without proof:

Theorem 0.1

The quadratic variation of an L^2-martingale, (X_t), is the unique (natural) increasing process,

(i) $\qquad \langle X \rangle_0 = 0 \quad \mathbb{P}$ a.s.

(ii) $\qquad (X_t^2 - \langle X \rangle_t)$ is an L^1 martingale.

Remark: We won't go into the details of what it means for a process to be natural. There is a discussion of this in an annex to these notes. Roughly speaking; there is only one natural increasing process, (A_t), such that $(X_t^2 - A_t)$ is an L^1-martingale. Since $(\langle X\rangle_t)$ is natural it is unique. So far as stochastic integration is concerned, this gives us a method for defining the integral: The steps are;

① For a simple process, $f(s) = \sum_{\theta} f_{t_{i-1}} I_{[t_{i-1}, t_i)}$, θ a partition of $[0, t]$,

$$\int_0^t f_s \, dX_s = \sum_{\theta} f_{t_{i-1}} \Delta X_{t_i}$$

② For a simple process,

$$\left\| \int_0^t f_s \, dX_s \right\|_2^2 = \mathbb{E}\left(\int_0^t f_s^2 \, d\langle X\rangle_s \right) \Big\} \text{ Isometry Property}$$

Stochastic Integral, Stieltjes Integral

③ Use ② to extend the integral to all (predictable) processes, f, for which

$$\mathbb{E}\left(\int_0^t f_s^2 \, d\langle X\rangle_s \right) < \infty, \quad t \in \mathbb{R}^+.$$

The integral exists as a limit in L^2-norm of sums of the form in ①.

When $X_t = \int_0^t g(s) \, dW_s$ then there is a "Radon-Nikodym" theorem; for suitable f,

$$\int_0^t f(s)\,dX_s = \int_0^t f(s)g(s)\,dW_s .$$

The quadratic variation, $(\langle X \rangle_t)$, of an L^2 martingale, has many of the features of a quadratic form. These crop up in many areas of mathematics. Often they are associated with a bilinear form which has the features of an inner product; indeed an example of a quadratic form and its associated bilinear form is the L^2 norm squared of $f \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ and the inner product $\langle f, g \rangle$, for f, g in L^2. The relationship between the quadratic form and the bilinear form, for this particular case, is

$$\langle f, g \rangle = \frac{1}{4}\left(\|f+g\|_2^2 - \|f-g\|_2^2 \right).$$

This is called the <u>polarisation identity</u>.

Given a quadratic form, the polarisation identity can be used to <u>define</u> a bilinear form. Applying this to the quadratic variation we define, for L^2 martingales, X, Y,

$$\langle X, Y \rangle_t = \frac{1}{4}\left(\langle X+Y \rangle_t - \langle X-Y \rangle_t \right)$$

The process $(\langle X, Y \rangle_t)$ is called the <u>cross-variation of X and Y</u>. In fact we can obtain the cross-variation as a limit over partitions, π, of $[0, T]$; Consider the sum

$$\sum_{\pi} \Delta X_{t_i} \Delta Y_{t_i} .$$

where $X = (X_t)$, $Y = (Y_t)$ are L^2-martingales. We know that $X+Y$ and $X-Y$ are L^2 martingales and that $\langle X+Y \rangle_T$ and $\langle X-Y \rangle_T$ exist for $T \in [0, \infty)$.

Observe that,

$$\Delta X_{t_i} \Delta Y_{t_i} = (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})$$

$$\Delta(X+Y)_{t_i} = (X_{t_i} + Y_{t_i}) - (X_{t_{i-1}} + Y_{t_{i-1}})$$
$$= \Delta X_{t_i} + \Delta Y_{t_i}$$

So $\Delta(X+Y)^2_{t_i} = \Delta X^2_{t_i} + 2\Delta X_{t_i} \Delta Y_{t_i} + \Delta Y^2_{t_i}$

and $\Delta(X-Y)^2_{t_i} = \Delta X^2_{t_i} - 2\Delta X_{t_i} \Delta Y_{t_i} + \Delta Y^2_{t_i}$

$$\Delta X_{t_i} \Delta Y_{t_i} = \frac{1}{4} \left\{ \Delta(X+Y)^2_{t_i} - \Delta(X-Y)^2_{t_i} \right\}$$

$$\sum_{\pi} \Delta X_{t_i} \Delta Y_{t_i} = \frac{1}{4} \left\{ \sum_{\pi} \Delta(X+Y)^2_{t_i} - \sum_{\pi} \Delta(X-Y)^2_{t_i} \right\}$$

Since the limit over π with $|\pi| \to 0$ exists for each of the sums,

$$\sum_{\pi} \Delta(X+Y)^2_{t_i} \quad \text{and} \quad \sum_{\pi} \Delta(X-Y)^2_{t_i}$$

then so does $\lim\limits_{\pi, |\pi| \to 0} \sum_{\pi} \Delta X_{t_i} \Delta Y_{t_i}$

and this is exactly $\langle X, Y \rangle_T$.

Recall Theorem 0.1; $\langle X \rangle$ is the (unique natural) increasing process such that $X^2 - \langle X \rangle$ is an L^1 martingale. This result extends to give us a characterisation of the cross variation :

First observe that

$$X_tY_t - \langle X,Y\rangle_t = X_tY_t - \frac{1}{4}\left\{\langle X+Y\rangle_t - \langle X-Y\rangle_t\right\}$$

Using 0.1, $(\langle X+Y\rangle)$ is the unique natural increas-ing process such^t that $(X+Y)^2 - \langle X+Y\rangle$ is an L^1 martingale: ditto, $(X-Y)^2 - \langle X-Y\rangle$ is an L^1 martingale. So, therefore, is

$$(X+Y)^2 - \langle X+Y\rangle \quad - \left((X-Y)^2 - \langle X-Y\rangle \right)$$

ie $\quad 4XY \quad - \left\{ \langle X+Y\rangle - \langle X-Y\rangle \right\}$

is an L^1 martingale, dividing by 4 tells us, $XY - \langle X,Y\rangle$ is an L^1 martingale. It turns out that $\langle X,Y\rangle$ is natural, too. An annex to these notes explains why this makes <u>$\langle X,Y\rangle$ the unique bounded variation process who's difference with XY is an L^1-marting-ale</u>.

The uniqueness of $\langle X,Y\rangle$ has many consequences:

(i) What is $\langle X,X\rangle$? Well, it is the unique natural bounded variation process such that $X^2 - \langle X,X\rangle$ is an L^1-martingale. This means $\langle X,X\rangle = \langle X\rangle$. See the annex for some details of the proof.

(ii) Let λ be a real number, what is $\langle \lambda X, Y\rangle$? Well, $XY - \langle X,Y\rangle$ is an L^1 martingale, so therefore is $\lambda(XY - \langle X,Y\rangle)$ and so $\lambda\langle X,Y\rangle$ is a (natural) bounded variation process whose difference with $(\lambda X)Y$ is an L^1-martingale.....

(iii) What is $\langle X+Y, Z\rangle$? We observe $(X+Y)Z = XZ + YZ$, and we know $XZ - \langle X,Z\rangle$ and $YZ - \langle Y,Z\rangle$ are L^1-martingales. So $XZ+YZ - (\langle X,Z\rangle + \langle Y,Z\rangle)$ is an L^1 martingale.....

So $\langle \cdot, \cdot \rangle$ is linear on the left....

(iv) $\langle Y, X\rangle$ is the unique natural.... such that $YX - \langle Y, X\rangle$ is an L^1-mart but $YX = XY$ and $XY - \langle X, Y\rangle$ is an L^1-mart....

So $\langle \cdot, \cdot \rangle$ is symmetric and therefore bilinear.

Stochastic Integrals and the Cross Variation.

Let X and Y be L^2 martingales and f_{t_0}, g_{t_0} bounded $\mathcal{F}_{t_0}$ measurable random variables. Let,

$$f(s) = f_{t_0} I_{[t_0, t_1)} \qquad g(s) = g_{t_0} I_{[t_0, t_1)}$$

then
$$\int_0^t f(s)\,dX_s = f_{t_0}(X_{t_1 \wedge t} - X_{t_0 \wedge t})$$
$$\int_0^t g(s)\,dY_s = g_{t_0}(Y_{t_1 \wedge t} - Y_{t_0 \wedge t}).$$

Evidently
$$\left(\int_0^t f(s)\,dX_s\right)\left(\int_0^t g(s)\,dY_s\right) = f_{t_0} g_{t_0} (X_{t_1\wedge t} - X_{t_0 \wedge t})(Y_{t_1 \wedge t} - Y_{t_0 \wedge t})$$

while $\int_0^t f(s)g(s)\,d\langle X,Y\rangle_s = f_{t_0}g_{t_0}(\langle X,Y\rangle_{t_1\wedge t} - \langle X,Y\rangle_{t_0\wedge t})$.

Therefore,

$$Z_t = \left(\int_0^t f\,dX\right)\left(\int_0^t g\,dY\right) - \int_0^t fg\,d\langle X,Y\rangle =$$

$$f_{t_0}g_{t_0}\big((X_{t_1\wedge t} - X_{t_0\wedge t})(Y_{t_1\wedge t} - Y_{t_0\wedge t}) - (\langle X,Y\rangle_{t_1\wedge t} - \langle X,Y\rangle_{t_0\wedge t})\big)$$

$$= f_{t_0}g_{t_0}\big((XY)_{t_1\wedge t} - X_{t_0\wedge t}Y_{t_1\wedge t} - X_{t_1\wedge t}Y_{t_0\wedge t} + (XY)_{t_0\wedge t} - \langle X,Y\rangle_{t_1\wedge t} + \langle X,Y\rangle_{t_0\wedge t}\big)$$

Now for $t \geq t_1$, $Z_t = Z_{t_1}$

and so for $t > r \geq t_1$ the left side is invariant under M_r and therefore $M_r(Z_t) = Z_r$. For $t_1 > t \geq t_0$ and $t > r \geq t_0$, $t_1 \wedge t = t$ and $M_r(Z_t)$ is:

$$M_r\big(f_{t_0}g_{t_0}((XY)_t - \langle X,Y\rangle_t - X_{t_0}Y_t - X_tY_{t_0} + (XY)_{t_0} + \langle X,Y\rangle_{t_0})\big),$$

as $(XY)_t - \langle X,Y\rangle_t$ is a martingale and $t_0 \leq r$ this is

$$= f_{t_0}g_{t_0}\big((XY)_r - \langle X,Y\rangle_r - X_{t_0}Y_r - X_rY_{t_0} + (XY)_{t_0} + \langle X,Y\rangle_{t_0} \big)$$

$$= f_{t_0}g_{t_0}\big((XY)_{t_1\wedge r} - \langle X,Y\rangle_{t_1\wedge t} - X_{t_0\wedge r}Y_{t_1\wedge r} - X_{t_1\wedge r}Y_{t_0\wedge r} + (XY)_{t_0\wedge r} + \langle X,Y\rangle_{t_0\wedge r}\big)$$

$$= Z_r.$$

Finally, if $r \leq t \leq t_0$ then 'everything is zero' and so $M_r(Z_t) = 0 = Z_r$.

So (Z_t) is an L^1-martingale! Also $\int_0^t fg\,d\langle X,Y\rangle$ is a bounded variation process since $\langle X,Y\rangle$ is a bounded variation process and f and g are bounded. We conclude that the process

$$\int_0^t f(s)g(s)\,d\langle X,Y\rangle_s \qquad t \geq 0$$

is the crossvariation of $\left(\int_0^t f\,dX\right)$ with $\left(\int_0^t g\,dY\right)$.

So this establishes an important result, but for elementary integrands only:

$$\left\langle \int_0^t f\,dX \;,\; \int_0^t g\,dY \right\rangle = \int_0^t fg\,d\langle X,Y\rangle$$

Memorise this result! "The cross-variation of the integrals is the integral of the product of the integrands with respect to the cross-variation of the integrators".

Without proof, we will assume that this result holds true for all "X-integrable f and Y-integrable g". This result is <u>very</u> useful.

Another very useful result is the multivariable form of Itô's Lemma: Here is a three variable form of it;

Write the preamble about Semimarts before Itô

Let N^1, N^2 be martingales and B^1, B^2, processes of bounded variation. Set, for $i=1,2$,

$$X_t^i = X_0^i + N_t^i + B_t^i$$

where X_0^i is $\mathcal{F}_0$ measurable. Let f be a $C^{1,2}$ function of three variables, t, x, y; $f = f(t,x,y)$.

$$f(t,X_t^1,X_t^2) = f(0,X_0^1,X_0^2) + \int_0^t \frac{\partial f}{\partial t}(s,X_s^1,X_s^2)\,ds$$

$\left(\text{this is } \int \frac{\partial f}{\partial x}\,dX^1\right)$ $$+ \int_0^t \frac{\partial f}{\partial x}(s,X_s^1,X_s^2)\,dM_s^1 + \int_0^t \frac{\partial f}{\partial x}(s,X_s^1,X_s^2)\,dB_s^1$$

$$\left(\text{this is } \int \frac{\partial f}{\partial y} dX^2\right) + \int_0^t \frac{\partial f}{\partial y}(s, X_s^1, X_s^2)\, dM_s^2 + \int_0^t \frac{\partial f}{\partial y}(s, X_s^1, X_s^2)\, dB_s^2$$

$$+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s^1, X_s^2)\, d\langle M^1\rangle_s + \int_0^t \frac{\partial^2 f}{\partial x \partial y}(s, X_s^1, X_s^2)\, d\langle M^1, M^2\rangle_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial y^2}(s, X_s^1, X_s^2)\, d\langle M^2\rangle_s$$

quad var — cross var — quad var

This line is the "second derivative term".

Before Itô

A Final result. The idea of quadratic variation can be extended to a wider class of processes than martingales. Indeed a process of the form of X^1 above, a martingale plus a bounded variation process(†), has a quadratic variation; if

$$X_t = X_0 + N_t + B_t$$

with N and B as above, then

$$\Delta X_{t_i} = \Delta N_{t_i} + \Delta B_{t_i}$$

and
$$\Delta X_{t_i}^2 = \Delta N_{t_i}^2 + 2\Delta N_{t_i}\Delta B_{t_i} + \Delta B_{t_i}^2$$

so that
$$\sum_\pi \Delta X_{t_i}^2 = \sum_\pi \Delta N_{t_i}^2 + 2\sum_\pi \Delta N_{t_i}\Delta B_{t_i} + \sum_\pi \Delta B_{t_i}^2$$

Since B is of bounded variation then along each path of B the quadratic variation of B is zero. Also, since we are dealing with continuous martingales, the term

$$\sum_\pi \Delta N_{t_i}\Delta B_{t_i}$$

tends to zero as the limit on π proceeds.

(†) We need B to be of integrable variation on $[0,t]$ for every $t \geq 0$.

This is because the paths of N are continuous. On $[0,t]$ these paths are uniformly continuous. As a consequence, when $|\pi|$ is sufficiently small,

$$\left|\sum_{\pi} \Delta N_{t_i}(\omega) \Delta B_{t_i}(\omega)\right| \leq \sum_{\pi} |\Delta N_{t_i}(\omega)| \cdot |\Delta B_{t_i}(\omega)|$$

This argument is not complete! Exercise!!

$$\leq \epsilon \sum_{\pi} |\Delta B_{t_i}(\omega)|$$

$$\leq \epsilon \operatorname{Var}_{[0,t]} B(\omega) .$$

So for processes of this form — semimartingales — the quadratic variation is just the quadratic variation of the martingale part, in short, $\langle X \rangle = \langle N \rangle$.

The general multidimensional version of Itô's Lemma — which you must learn — is as follows;

Let $X^i = X^i_0 + N^i + A^i$, $1 \leq i \leq n$, be semimartingales with X^i_0 an $\mathcal{F}_0$ measurable random variable, N^i a martingale and A^i a bounded variation process. Let $f(t, \underline{x})$ be a $C^{1,2}$ function from $[0,\infty) \times \mathbb{R}^n$ into $\mathbb{R}$. Then writing $\underline{X}_t = (X^1_t, X^2_t, X^3_t, \ldots, X^n_t)$

$$f(t, \underline{X}_t) = f(0, \underline{X}_0) + \int_0^t \frac{\partial f}{\partial t}(s, \underline{X}_s)\, ds + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x^i}(s, \underline{X}_s)\, dN^i_s$$

$$+ \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(s, \underline{X}_s)\, dA^i_s \qquad +$$

$$+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(s, \underline{X}_s)\, d\langle N^i, N^j \rangle_s .$$

Notice that $\int_0^t \frac{\partial f}{\partial x^i}(s,X_s)\,dN_s^i + \int_0^t \frac{\partial f}{\partial x^i}(s,X_s)\,dA_s^i = \int_0^t \frac{\partial f}{\partial x_i}(s,X_s)\,dX_s^i$.

The two dimensional version without any dependence upon time, t, is obtained simply by setting '$\frac{\partial f}{\partial t} \equiv 0$.' See the exercises for some examples.

CB Write the 3 variable case here in the semi-mart form.

Exercise: Prove the Semi-martingale product rule.

(i) $$X_t Y_t = X_0 Y_0 + \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + \langle X, Y\rangle_t$$

(ii) Let $$S_t = S_0 + \int_0^t \mu^S S_s\,ds + \int_0^t \sigma^S S_s\,dW_s^S \quad \text{Foreign Value}$$

$$F_t = F_0 + \int_0^t \mu^f F_s\,ds + \int_0^t \sigma^f F_s\,dW_s^f \quad \text{Exchange Rate}$$

We assume $\langle W^f, W^S\rangle_t = \rho t$. The domestic value of S is SF. Write down the stochastic equation for SF.